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STRETCH AND TWIST OF ROTATING PRISMATIC BEAM

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Abstract—A turbine blade is modelled as a uniform isotropic prismatic beam of general cross-section and "setting angle" rotating about one end, and is analysed according to the linear theory of elasticity. A semi-inverse solution is presented which reduces the three-dimensional problem to one of two dimensions, and explicit stress and strain components given for the mathematically amenable elliptic cross-section. As expected, the planar stresses σ_x, σ_y and τ_{xy} arising from the two-dimensional problem are found to be small. For the general section, the theory predicts small curvature of the blade centre line, and a twisting moment which tends to reduce the "angle of set".

1. INTRODUCTION

The determination of stress and strain fields in prismatic rods of arbitrary cross-section due to forces applied at the ends of the rod only, is known as Saint-Venant's problem. Solutions have been obtained for tension, pure bending, torsion and bending due to a terminal shearing force (see, e.g. Love[1]); apart from their direct application, these solutions provide justification and limitations to the technical theories used in "strength of materials". Exact solutions have also been obtained for body force gravity loadings of rods and beams producing longitudinal extension (see Sokolnikoff[2], Chap. 4) and bending (see Love[1], Chap. 16). In principle, exact solutions can be obtained for any case in which the forces applied to the beam along its length can be represented by rational integral functions of the beam axial coordinate[3].

In this paper, the authors present a semi-inverse solution for the centrifugal body force loading in a prismatic beam rotating about any axis through one end perpendicular to the longitudinal centroidal axis, as shown in Fig. 1, where the section principal axes (x', y') are inclined to the axis of rotation and tangential direction (x, y) with a setting angle β; the z-axis coincides with the beam centroidal axis.

For a general asymmetric cross-section the theory predicts curvature of the blade centre line; while this effect is small, the authors have found no previous reference in the literature. When the x-axis of rotation is not a principal axis of the section, there is also a twisting moment, which for an airfoil section would tend to turn the chord into the plane of rotation thereby reducing the angle of set.

A complete solution for the mathematically amenable elliptic cross-section is given, from which the relative magnitudes of the various stress components are compared; as expected the planar σ_x, σ_y and τ_{xy} components are found to be small, as is variation of the longitudinal stress σ_z over the cross-section.

As is usual, the derived solution satisfies the boundary conditions on the surface generators of the beam; over the free end, the stress free condition is satisfied on a macroscopic level by requiring the stress resultants to be zero. Here we appeal to Saint-Venant's principle and argue, as at the root section, that these shortcomings will produce only local differences.

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Compatibility equations, eqns (3a)–(3c), become

$$\left.\begin{aligned}\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}&=\frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y}\\ \frac{\partial^2 \varepsilon_y}{\partial z^2}+\frac{\partial^2 \varepsilon_z}{\partial y^2}&=\theta_1\frac{\partial^2 \phi}{\partial y^2}\\ \frac{\partial^2 \varepsilon_z}{\partial x^2}+\frac{\partial^2 \varepsilon_x}{\partial z^2}&=\theta_1\frac{\partial^2 \phi}{\partial x^2}\end{aligned}\right\} \qquad (10\text{a–c})$$

and eqns (4a)–(4c) become

$$\left.\begin{aligned}\frac{\partial^2 \varepsilon_x}{\partial y\,\partial z}&=0\\ \frac{\partial^2 \varepsilon_y}{\partial x\,\partial z}&=0\\ \frac{\partial^2 \varepsilon_z}{\partial x\,\partial y}&=\theta_1\frac{\partial^2 \phi}{\partial x\,\partial y}.\end{aligned}\right\} \qquad (11\text{a–c})$$

Integration of eqn (11c) and comparison with eqn (7) requires

$$f(x,y)=\theta_1\phi+f_1(x)+f_2(y).$$

Now from Hooke's laws (5c)

$$\varepsilon_z=\frac{\sigma_z}{E}-\frac{\nu}{E}(\sigma_x+\sigma_y)$$

when differentiated with respect to z, gives

$$\frac{\partial\sigma_z}{\partial z}=E\frac{\partial\varepsilon_z}{\partial z}$$

since

$$\frac{\partial\sigma_x}{\partial z}=\frac{\partial\sigma_y}{\partial z}=0$$

and from eqn (7) we have

$$\frac{\partial\sigma_z}{\partial z}=E\frac{\partial\varepsilon_z}{\partial z}=-\rho\Omega^2 z.$$

Thus the equilibrium equation, eqn (8c), requires $\nabla^2\phi=0$.

Substituting the assumed shear stresses τ_{xz} and τ_{yz} into boundary condition (2c) it is evident that ϕ is the Saint-Venant torsion function.

Hooke's laws, eqns (5a) and (5b), differentiated twice with respect to z, give

$$\frac{\partial^2\varepsilon_x}{\partial z^2}=\frac{\partial^2\varepsilon_y}{\partial z^2}=-\frac{\nu}{E}\frac{\partial^2\sigma_z}{\partial z^2}=-\nu\frac{\partial^2\varepsilon_z}{\partial z^2}=\frac{\nu\rho\Omega^2}{E}.$$

Hence compatibility equations, eqns (10b) and (10c), become

$$\left.\begin{aligned}\frac{\partial^2 \varepsilon_z}{\partial x^2} &= \theta_1 \frac{\partial^2 \phi}{\partial x^2} - \frac{\nu\rho\Omega^2}{E}\\ \frac{\partial^2 \varepsilon_z}{\partial y^2} &= \theta_1 \frac{\partial^2 \phi}{\partial y^2} - \frac{\nu\rho\Omega^2}{E}\end{aligned}\right\}$$

but from eqn (7) we have

$$\left.\begin{aligned}\frac{\partial^2 \varepsilon_z}{\partial x^2} &= 2\varepsilon_1 + \theta_1 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 f_1(x)}{\partial x^2}\\ &\text{and}\\ \frac{\partial^2 \varepsilon_z}{\partial y^2} &= 2\varepsilon_1 + \theta_1 \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 f_2(y)}{\partial y^2}\end{aligned}\right\}$$

from which we conclude

$$\varepsilon_1 = -\frac{\nu\rho\Omega^2}{2E}, \qquad \frac{\partial^2 f_1(x)}{\partial x^2} = \frac{\partial^2 f_2(y)}{\partial y^2} = 0$$

or

$$f_1(x) = ax + b, \qquad f_2(y) = cy + d.$$

Since we already have constant and linear terms in ε_z, which are as yet undetermined, the above are incorporated, and we put $f_1(x) = f_2(y) = 0$.

The longitudinal stress is now written as

$$\sigma_z = E\varepsilon_0 + \frac{\rho\Omega^2}{2}(L^2 - z^2) - E\kappa_0 x - E\kappa_0' y - \frac{\nu\rho\Omega^2}{2}(x^2 + y^2) + E\theta_1\phi + \nu(\sigma_x + \sigma_y).$$

To evaluate the constant ε_0, we construct the tensile force

$$T = \iint \sigma_z \, \mathrm{d}x \, \mathrm{d}y \; EA\varepsilon_0 + \frac{\rho A\Omega^2}{2}(L^2 - z^2) - \frac{\nu\rho\Omega^2}{2}(I_x + I_y) + E\theta_1 \iint \phi \, \mathrm{d}x \, \mathrm{d}y$$
$$+ \nu \iint (\sigma_x + \sigma_y) \, \mathrm{d}x \, \mathrm{d}y.$$

Now the last integral may be expressed as

$$\iint (\sigma_x + \sigma_y) \, \mathrm{d}x \, \mathrm{d}y = \iint \left\{ \frac{\partial}{\partial x}(x\sigma_x + y\tau_{xy}) + \frac{\partial}{\partial y}(x\tau_{xy} + y\sigma_y) \right\} \mathrm{d}x \, \mathrm{d}y$$
$$- \iint \left\{ x\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) + y\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right) \right\} \mathrm{d}x \, \mathrm{d}y.$$

The first integral transforms to the line integral

$$\int_c \{x(\sigma_x l+\tau_{xy}m)+y(\tau_{xy}l+\sigma_y m)\}\, ds$$

which is zero by virtue of boundary conditions (2a) and (2b). Employing the equilibrium equations, eqns (8a) and (8b), we have from the second integral

$$\iint(\sigma_x+\sigma_y)\, dx\, dy = \iint\left\{G\theta_1 x\left(\frac{\partial\phi}{\partial x}-y\right)+G\theta_1 y\left(\frac{\partial\phi}{\partial y}+x\right)+\rho\Omega^2 y^2\right\} dx\, dy$$

$$= G\theta_1\iint\left(x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}\right) dx\, dy+\rho\Omega^2 I_x.$$

Now introduce the conjugate function ψ via the Cauchy–Riemann equations

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y},\qquad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

and

$$\iint\left(x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}\right) dx\, dy = \iint\left\{\frac{\partial}{\partial y}(x\psi)-\frac{\partial}{\partial x}(y\psi)\right\} dx\, dy$$

$$= -\int_c\{(x\psi)\, dx+(y\psi)\, dy\}$$

$$= -\int_c\left\{\left(\frac{x^2+y^2}{2}\right)x\, dx+\left(\frac{x^2+y^2}{2}\right)y\, dy\right\}$$

since $\psi = \frac{1}{2}(x^2+y^2)$ on the boundary.

Transforming back to an area integral the above becomes

$$\iint\left\{\frac{\partial}{\partial y}\left(\left(\frac{x^2+y^2}{2}\right)x\right)-\frac{\partial}{\partial x}\left(\left(\frac{x^2+y^2}{2}\right)y\right)\right\} dx\, dy = 0$$

hence

$$\iint(\sigma_x+\sigma_y)\, dx\, dy = \rho\Omega^2 I_x.$$

We also write

$$c = -\frac{1}{A}\iint\phi\, dx\, dy \tag{12}$$

from which the tensile force may be written as

$$T=\frac{\rho A\Omega^2}{2}(L^2-z^2)+EA\varepsilon_0-EAc\theta_1+\frac{\nu\rho\Omega^2}{2}(I_x-I_y).$$

Now we require the tensile force to be zero at the free end $z = L$ which gives

$$\varepsilon_0 = c\theta_1 - \frac{\nu\rho\Omega^2}{2EA}(I_x - I_y)$$

and hence

$$T = \frac{\rho A\Omega^2}{2}(L^2 - z^2) \tag{13}$$

as expected from elementary theory.

To evaluate the constants κ_0 and κ'_0 we construct the bending moments

$$M_x = \iint y\sigma_z \,\mathrm{d}x\,\mathrm{d}y, \qquad M_y = -\iint x\sigma_z \,\mathrm{d}x\,\mathrm{d}y.$$

Now

$$M_x = \iint \left\{ E\varepsilon_0 y + \frac{\rho\Omega^2}{2}(L^2 - z^2)y - E\kappa_0 xy - E\kappa'_0 y^2 - \frac{\nu\rho\Omega^2}{2}(x^2 + y^2)y \right.$$
$$\left. + E\theta_1\phi y + \nu(\sigma_x + \sigma_y)y \right\} \mathrm{d}x\,\mathrm{d}y$$

or

$$M_x = -E\kappa_0 I_{xy} - E\kappa'_0 I_x - \frac{\nu\rho\Omega^2}{2}\iint (x^2 + y^2)y \,\mathrm{d}x\,\mathrm{d}y$$
$$+ E\theta_1 \iint \phi y \,\mathrm{d}x\,\mathrm{d}y + \nu \iint (\sigma_x + \sigma_y)y \,\mathrm{d}x\,\mathrm{d}y.$$

Now the last integral may be written as

$$\iint (\sigma_x + \sigma_y)y \,\mathrm{d}x\,\mathrm{d}y = \iint \left\{ \frac{\partial}{\partial y}\left(\left(\frac{y^2 - x^2}{2}\right)\sigma_y + xy\tau_{xy} \right) + \frac{\partial}{\partial x}\left(xy\sigma_x + \left(\frac{y^2 - x^2}{2}\right)\tau_{xy} \right) \right\} \mathrm{d}x\,\mathrm{d}y$$
$$- \iint \left\{ xy\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} \right) + \left(\frac{y^2 - x^2}{2}\right)\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} \right) \right\} \mathrm{d}x\,\mathrm{d}y;$$

the first integral transforms to the line integral

$$\int_c \left\{ xy(\sigma_x l + \tau_{xy} m) + \left(\frac{y^2 - x^2}{2}\right)(\tau_{xy} l + \sigma_y m) \right\} \mathrm{d}s$$

which is zero by virtue of boundary conditions (2a) and (2b), whilst the second integral yields, from eqns (8a) and (8b)

$$\iint (\sigma_x + \sigma_y)y \,\mathrm{d}x\,\mathrm{d}y = \frac{\rho\Omega^2}{2}\iint (y^2 - x^2)y \,\mathrm{d}x\,\mathrm{d}y - \frac{G\theta_1}{2}\iint (x^2 + y^2)x \,\mathrm{d}x\,\mathrm{d}y$$
$$+ G\theta_1 \iint \left\{ xy\frac{\partial\phi}{\partial x} + \left(\frac{y^2 - x^2}{2}\right)\frac{\partial\phi}{\partial y} \right\} \mathrm{d}x\,\mathrm{d}y.$$

Again introducing the conjugate function ψ via the Cauchy–Riemann equations, and the Prandtl stress function Ψ through the relationship

$$\psi = \Psi + \left(\frac{x^2+y^2}{2}\right)$$

and after some manipulation we find

$$\iint(\sigma_x+\sigma_y)y\,\mathrm{d}x\,\mathrm{d}y = \frac{\rho\Omega^2}{2}\iint(y^2-x^2)y\,\mathrm{d}x\,\mathrm{d}y - 2G\theta_1\iint x\Psi\,\mathrm{d}x\,\mathrm{d}y$$

and hence we have for the bending moment

$$M_x = -EI_{xy}\kappa_0 - EI_x\kappa_0' - \nu\rho\Omega^2\iint x^2y\,\mathrm{d}x\,\mathrm{d}y + E\theta_1\left[\iint\phi y\,\mathrm{d}x\,\mathrm{d}y - \frac{\nu}{1+\nu}\iint x\Psi\,\mathrm{d}x\,\mathrm{d}y\right].$$

Similarly we find for the bending moment

$$M_y = E\kappa_0 I_y + E\kappa_0' I_{xy} + \frac{\nu\rho\Omega^2}{2}\iint(x^2+y^2)x\,\mathrm{d}x\,\mathrm{d}y - E\theta_1\iint\phi x\,\mathrm{d}x\,\mathrm{d}y$$
$$-\nu\iint(\sigma_x+\sigma_y)x\,\mathrm{d}x\,\mathrm{d}y$$

and with

$$\iint(\sigma_x+\sigma_y)x\,\mathrm{d}x\,\mathrm{d}y = \rho\Omega^2\iint xy^2\,\mathrm{d}x\,\mathrm{d}y + 2G\theta_1\iint y\Psi\,\mathrm{d}x\,\mathrm{d}y$$

the moment becomes

$$M_y = E\kappa_0 I_y + E\kappa_0' I_{xy} + \frac{\nu\rho\Omega^2}{2}\iint(x^2-y^2)x\,\mathrm{d}x\,\mathrm{d}y$$
$$-E\theta_1\left[\iint\phi x\,\mathrm{d}x\,\mathrm{d}y + \frac{\nu}{1+\nu}\iint y\Psi\,\mathrm{d}x\,\mathrm{d}y\right].$$

Now since the bending moments M_x and M_y are seen to be independent of the axial coordinate z, and are known to be zero at the tip $z = L$ we must have $M_x = M_y = 0$; this allows the constants κ_0 and κ_0' (curvatures) to be written as

$$\kappa_0 = \theta_1\left[\iint\left(\frac{I_x x}{D} - \frac{I_{xy}y}{D}\right)\phi\,\mathrm{d}x\,\mathrm{d}y + \frac{\nu}{1+\nu}\iint\left(\frac{I_{xy}x}{D} + \frac{I_x y}{D}\right)\Psi\,\mathrm{d}x\,\mathrm{d}y\right]$$
$$-\frac{\nu\rho\Omega^2}{E}\left[\frac{I_x}{D}\iint\left(\frac{x^2-y^2}{2}\right)x\,\mathrm{d}x\,\mathrm{d}y - \frac{I_{xy}}{D}\iint x^2y\,\mathrm{d}x\,\mathrm{d}y\right]$$

$$\kappa_0' = -\theta_1\left[\iint\left(\frac{I_{xy}x}{D} - \frac{I_y y}{D}\right)\phi\,\mathrm{d}x\,\mathrm{d}y + \frac{\nu}{1+\nu}\iint\left(\frac{I_y x}{D} + \frac{I_{xy}y}{D}\right)\Psi\,\mathrm{d}x\,\mathrm{d}y\right]$$
$$-\frac{\nu\rho\Omega^2}{E}\left[\frac{I_y}{D}\iint x^2y\,\mathrm{d}x\,\mathrm{d}y - \frac{I_{xy}}{D}\iint\left(\frac{x^2-y^2}{2}\right)x\,\mathrm{d}x\,\mathrm{d}y\right] \quad \text{(14a, b)}$$

where $D = I_xI_y - I_{xy}^2$.

Now the θ_1 coefficients in eqns (14) can be recognized as the centre of flexure (shear centre) coordinates for a centroidal non-principal coordinate system[4], i.e.

$$x_F = \iint\left(\frac{I_{xy}x}{D} - \frac{I_y y}{D}\right)\phi \, dx \, dy + \frac{\nu}{1+\nu}\iint\left(\frac{I_y x}{D} + \frac{I_{xy}y}{D}\right)\Psi \, dx \, dy$$

$$y_F = \iint\left(\frac{I_x x}{D} - \frac{I_{xy}y}{D}\right)\phi \, dx \, dy + \frac{\nu}{1+\nu}\iint\left(\frac{I_{xy}x}{D} + \frac{I_x y}{D}\right)\Psi \, dx \, dy \qquad (15a, b)$$

and the expressions for the curvatures reduce to

$$\kappa_0 = y_F\theta_1 - \frac{\nu\rho\Omega^2}{E}\left[\frac{I_x}{D}\iint\frac{(x^2-y^2)}{2}x \, dx \, dy - \frac{I_{xy}}{D}\iint x^2 y \, dx \, dy\right]$$

$$\kappa_0' = -x_F\theta_1 - \frac{\nu\rho\Omega^2}{E}\left[\frac{I_y}{D}\iint x^2 y \, dx \, dy - \frac{I_{xy}}{D}\iint\frac{(x^2-y^2)}{2}x \, dx \, dy\right]. \qquad (16a, b)$$

To evaluate the constant θ_1, we construct the twisting moment

$$M_z = \iint(x\tau_{yz} - y\tau_{xz}) \, dx \, dy$$

and hence the rate of change of twisting moment

$$\frac{\partial M_z}{\partial z} = \frac{\partial}{\partial z}\left\{\iint(x\tau_{yz} - y\tau_{xz}) \, dx \, dy\right\}$$

which is evaluated in two ways. Firstly we employ eqns (6) to find

$$\frac{\partial M_z}{\partial z} = G\theta_1\iint\left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} + x^2 + y^2\right) dx \, dy$$

or

$$\frac{\partial M_z}{\partial z} = GJ\theta_1$$

where

$$J = \iint\left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} + x^2 + y^2\right) dx \, dy$$

is the torsion constant.

Secondly we employ the equilibrium equations, eqns (1a) and (1b), to obtain

$$\frac{\partial M_z}{\partial z} = -\rho\Omega^2 \iint xy\,\mathrm{d}x\,\mathrm{d}y + \iint \left\{ y\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) - x\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right)\right\} \mathrm{d}x\,\mathrm{d}y.$$

The second area integral may be transformed to the line integral

$$\int_c \{y(\sigma_x l + \tau_{xy} m) - x(\tau_{xy} l + \sigma_y m)\}\,\mathrm{d}s$$

which is zero by virtue of the boundary conditions, hence

$$\frac{\partial M_z}{\partial z} = -\rho\Omega^2 I_{xy} = GJ\theta_1$$

and

$$\theta_1 = -\frac{\rho\Omega^2 I_{xy}}{GJ}. \tag{17}$$

Now since the twisting moment M_z is zero at the tip $z = L$, we find $M_z = \rho\Omega^2 I_{xy}(L-z)$.

Before consideration of the two-dimensional problem for any particular cross-section we now summarize the stress and strain components as

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} - G\theta_1(\phi - xy)$$

$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2} - G\theta_1(\phi + xy) - \frac{\rho\Omega^2 y^2}{2}$$

$$\sigma_z = \frac{\rho\Omega^2}{2}(L^2 - z^2) - E\kappa_0 x - E\kappa'_0 y + \frac{\nu\rho\Omega^2}{2}\left[\frac{I_y - I_x}{A} - (x^2+y^2)\right]$$
$$+\nu\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} - \frac{\rho\Omega^2 y^2}{2}\right) + E\theta_1 c + 2G\theta_1\phi$$

$$\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x\,\partial y}$$

$$\tau_{xz} = G\theta_1(z-L)\left(\frac{\partial \phi}{\partial x} - y\right) = \frac{\rho\Omega^2 I_{xy}}{J}(L-z)\left(\frac{\partial \phi}{\partial x} - y\right)$$

$$\tau_{yz} = G\theta_1(z-L)\left(\frac{\partial \phi}{\partial y} + x\right) = \frac{\rho\Omega^2 I_{xy}}{J}(L-z)\left(\frac{\partial \phi}{\partial y} + x\right)$$

$$\varepsilon_x = -\frac{\nu\rho\Omega^2}{2E}(L^2 - z^2) + \nu\kappa_0 x + \nu\kappa'_0 y - \frac{\nu^2\rho\Omega^2}{2E}\left(\frac{(I_y - I_x)}{A} - (x^2+y^2)\right)$$
$$+\frac{(1-\nu^2)}{E}\frac{\partial^2 \Phi}{\partial y^2} - \frac{\nu(1+\nu)}{E}\left(\frac{\partial^2 \Phi}{\partial x^2} - \frac{\rho\Omega^2 y^2}{2}\right) + \frac{\theta_1 xy}{2} - \nu\theta_1 c - \frac{\theta_1\phi}{2}$$

$$\varepsilon_y = -\frac{\nu\rho\Omega^2}{2E}(L^2 - z^2) + \nu\kappa_0 x + \nu\kappa'_0 y - \frac{\nu^2\rho\Omega^2}{2E}\left(\frac{I_y - I_x}{A} - (x^2+y^2)\right)$$
$$+\frac{(1-\nu^2)}{E}\left(\frac{\partial^2 \Phi}{\partial x^2} - \frac{\rho\Omega^2 y^2}{2}\right) - \frac{\nu(1+\nu)}{E}\frac{\partial^2 \Phi}{\partial y^2} - \frac{\theta_1 xy}{2} - \nu\theta_1 c - \frac{\theta_1\phi}{2}$$

$$\varepsilon_z = \frac{\rho\Omega^2}{2E}(L^2-z^2)-\kappa_0 x-\kappa_0' y+\frac{\nu\rho\Omega^2}{2E}\left[\frac{(I_y-I_x)}{A}-(x^2+y^2)\right]+\theta_1(\phi+c)$$

$$\gamma_{xy} = -\frac{1}{G}\frac{\partial^2\Phi}{\partial x\,\partial y}$$

$$\gamma_{xz} = \theta_1(z-L)\left(\frac{\partial\phi}{\partial x}-y\right) = \frac{\rho\Omega^2 I_{xy}}{GJ}(L-z)\left(\frac{\partial\phi}{\partial x}-y\right)$$

$$\gamma_{yz} = \theta_1(z-L)\left(\frac{\partial\phi}{\partial y}+x\right) = \frac{\rho\Omega^2 I_{xy}}{GJ}(L-z)\left(\frac{\partial\phi}{\partial y}+x\right). \quad (18a\text{–}l)$$

2.3. *Two-dimensional problem*

The compatibility equation, eqn (9a), yields the biharmonic equation

$$\nabla^4\Phi = -\left(\frac{\nu+3\nu^2}{1-\nu^2}\right)\rho\Omega^2. \quad (19)$$

Boundary conditions (2a) and (2b) become, on putting $l = \mathrm{d}y/\mathrm{d}s$, $m = -\mathrm{d}x/\mathrm{d}s$

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\Phi}{\partial y}\right) &= G\theta_1(\phi-xy)\frac{\mathrm{d}y}{\mathrm{d}s}\\ \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\Phi}{\partial x}\right) &= \left(\frac{\rho\Omega^2 y^2}{2}+G\theta_1(\phi+xy)\right)\frac{\mathrm{d}x}{\mathrm{d}s}.\end{aligned}\right\} \quad (20)$$

3. EXAMPLE—ELLIPTIC CROSS-SECTION BEAM

3.1.

We consider the ellipse $x'^2/a^2+y'^2/b^2-1 = 0$, which has principal axes inclined at setting angle β. The boundary equation may be written as

$$B(x,y) = c_1x^2+c_2y^2+c_3xy-a^2b^2 = 0$$

where

$$c_1 = a^2\sin^2\beta+b^2\cos^2\beta$$
$$c_2 = a^2\cos^2\beta+b^2\sin^2\beta$$
$$c_3 = 2(b^2-a^2)\sin\beta\cos\beta.$$

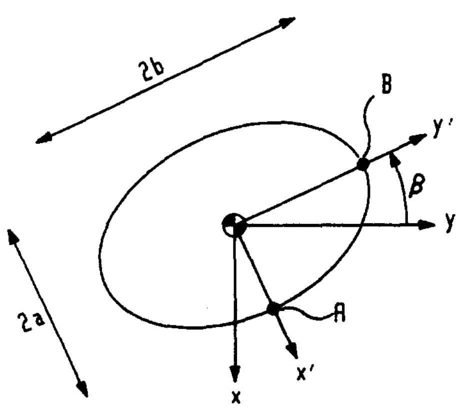

Fig. 2. Elliptic cross-section beam with arbitrary "angle of set" β.

The Saint-Venant torsion function

$$\phi(x', y') = -\left(\frac{a^2-b^2}{a^2+b^2}\right)x'y'$$

becomes

$$\phi(x,y) = -\left(\frac{a^2-b^2}{a^2+b^2}\right)[(y^2-x^2)\sin\beta\cos\beta + xy(\cos^2\beta - \sin^2\beta)] \tag{21}$$

and for the stress function we find

$$\Phi(x,y) = \frac{\Lambda}{8}\Delta B^2(x,y) - \frac{\rho\Omega^2}{24a^2b^2}c_3B(x,y)xy + \frac{\rho\Omega^2}{12}c_3xy + \frac{\rho\Omega^2}{4}c_1x^2 + \frac{\rho\Omega^2}{192a^2b^2}c_3^2(y^4 - x^4 + 2x^2y^2) - \frac{\rho\Omega^2}{24a^2b^2}c_1^2x^4 \tag{22}$$

and hence stresses

$$\sigma_x = \frac{\Lambda}{4}\Delta[(2c_2y + c_3x)^2 + 2c_2B(x,y)]$$

$$\sigma_y = \frac{\Lambda}{4}\Delta[(2c_1x + c_3y)^2 + 2c_1B(x,y)] - \frac{\rho\Omega^2}{2a^2b^2}c_1B(x,y)$$

$$\sigma_z = \frac{\rho\Omega^2}{2}(L^2 - z^2) + \frac{v\rho\Omega^2}{2}\left(\frac{(c_2-c_1)}{4} - (x^2+y^2)\right) + \frac{(1+v)(a^2+b^2)}{4a^2b^2}\rho\Omega^2c_3\phi(x,y) + \frac{v\Lambda}{4}\Delta((2c_2y + c_3x)^2 + (2c_1x + c_3y)^2) + \frac{v\Lambda}{2}\Delta(c_1+c_2)B(x,y) - \frac{v\rho\Omega^2}{2a^2b^2}c_1B(x,y)$$

$$\tau_{xy} = -\frac{\Lambda}{4}\Delta[(2c_1x + c_3y)(2c_2y + c_3x) + c_3B(x,y)] + \frac{\rho\Omega^2c_3}{8a^2b^2}B(x,y)$$

$$\tau_{xz} = \frac{\rho\Omega^2}{8a^2b^2}c_3(L-z)(2c_2y + c_3x)$$

$$\tau_{yz} = -\frac{\rho\Omega^2}{8a^2b^2}c_3(L-z)(2c_1x + c_3y) \tag{23a–f}$$

where

$$\Lambda = \frac{\rho\Omega^2}{a^2b^2(3a^4 + 2a^2b^2 + 3b^4)}$$

$$\Delta = \left(a^4\sin^2\beta + b^4\cos^2\beta - \frac{v+3v^2}{1-v^2}a^2b^2\right).$$

Table 1

Setting angle β (deg.)	Location	Stress magnitudes					
		$\sigma_x/\bar{\sigma}_z \times 100\%$	$\sigma_y/\bar{\sigma}_z \times 100\%$	$\tau_{xy}/\bar{\sigma}_z \times 100\%$	$\sigma_z/\bar{\sigma}_z \times 100\%$	$\tau_{xz}/\bar{\sigma}_z \times 100\%$	$\tau_{yz}/\bar{\sigma}_z \times 100\%$
0	A	0	3.96	0	100.66	0	0
	B	0.16	0	0	97.72	0	0
30	A	0.74	2.20	−1.27	100.58	26.0	−45.0
	B	0.08	0.03	0.05	97.9	9.0	5.2
45	A	0.96	0.96	−0.96	100.50	42.4	−42.4
	B	0.04	0.04	0.04	98.15	8.5	8.5
90	A	−0.09	0	0	100.30	0	0
	B	0	−0.004	0	98.60	0	0

3.2. *Comparison of stress magnitudes*

Stresses have been calculated at several locations on the boundary surface of the ellipse with various setting angles β, and comparison made with the average of the longitudinal stress σ_z at the root $z = L$, i.e. $\bar{\sigma}_z = \rho\Omega^2L^2/2$. A selection of these results is shown in Table 1 above.

Stresses σ_z, τ_{xz} and τ_{yz} all have their maximum values at the root $z = 0$, and are all zero at the tip $z = L$; since σ_z decreases with z^2, whereas τ_{xz} and τ_{yz} decrease linearly with z, the above values of $\tau_{xz}/\bar{\sigma}_z$ and $\tau_{yz}/\bar{\sigma}_z$ will be lower at other sections along the beam, i.e. $L > z > 0$. Clearly τ_{xz} and τ_{yz} are of considerable magnitude and, as expected, have maximum values on the surface closest to the centroid.

The planar stresses σ_x, σ_y and τ_{xy} are independent of z, and thus would assume greater proportion of σ_z for $L > z > 0$; again the maximum boundary value appears to occur closest to the centroid.

Due to the symmetry of the section in this example, the centre of flexure coincides with the centroid, and the curvatures κ_0 and κ_0' are zero; thus the variation in σ_z over the root section indicated by $\sigma_z/\bar{\sigma}_z$ appears to be caused by the distortion of the initially plane cross-section into a paraboloid of revolution together with a contribution from $\nu\sigma_x$ and $\nu\sigma_y$. For an asymmetric section involving bending, it is thought that the variation in σ_z would be greater.

4. CONCLUSION

The present theory provides an exact linear elasticity solution for the centrifugal body force loading of a prismatic isotropic beam of general cross-section rotating about one end; complete determination of stress and strain components requires a knowledge of the Saint-Venant torsion function and solution of a two-dimensional biharmonic boundary value problem for the section. The theory predicts extensional, bending and torsional stresses and displacements; for a beam of elliptic cross-section the in-plane stresses (σ_x, σ_y and τ_{xy}) are found to be small.

Apart from classical interest it is thought the theory will supplement computational work by providing an exact solution for a simple turbine blade model for the purpose of validation and understanding of numerical predictions.

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